

The Binomial Coefficient for Negative Arguments

M.J. Kronenburg

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Abstract

The definition of the binomial coefficient in terms of gamma functions also allows non-integer arguments. For nonnegative integer arguments the gamma functions reduce to factorials, leading to the well-known Pascal triangle. Using a symmetry formula for the gamma function, this definition is extended to negative integer arguments, making it continuous at all integer arguments. The agreement of this definition with some other identities and with the binomial theorem is investigated.

Keywords: binomial coefficient, gamma function.

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1 Binomial Coefficients and the Gamma Function

The definition of the binomial coefficient in terms of gamma functions for complex x , y is [1]:

$$\binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)} \quad (1.1)$$

For nonnegative integer n and integer k this reduces to [1]:

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

The value of the binomial coefficient (1.1) for negative integer n and integer k can be computed using the following symmetry formula for the gamma function.

Theorem 1.1. *For integer a , b and complex s :*

$$\frac{\Gamma(s-a+1)}{\Gamma(s-b+1)} = (-1)^{b-a} \frac{\Gamma(b-s)}{\Gamma(a-s)} \quad (1.3)$$

Proof. The reflection formula for the gamma function is [1, 2]:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad (1.4)$$

Combination of this formula with a special case of the addition formula for the sine function [1, 2]:

$$\sin(\pi(b-s)) = (-1)^{b-a} \sin(\pi(a-s)) \quad (1.5)$$

yields the result. \square

The recursion formula for the gamma function, which is also used below, is for complex s [1]:

$$\Gamma(s+1) = s \Gamma(s) \quad (1.6)$$

2 Binomial Coefficient for Negative First Argument

When a gamma function in the definition of the binomial coefficient (1.1) has non-positive integer argument, the value of that gamma function is infinite. The binomial coefficient (1.1) for nonnegative integer x and integer y can have at most one denominator gamma function with nonpositive integer argument, in which case the binomial coefficient becomes zero, and this is expressed by identity (1.2). For negative integer x and integer y the numerator and at least one of the denominator gamma functions have nonpositive integer arguments. In this case (1.3) with $s = 0$ can be used to replace two of these infinite gamma functions with finite ones, which yields computable expressions for all integers. This results in the following binomial coefficient identity, which with identity (1.2) allows computation of the binomial coefficient for all integer arguments.

Theorem 2.1. *For negative integer n and integer k :*

$$\binom{n}{k} = \begin{cases} (-1)^k \binom{-n+k-1}{k} & \text{if } k \geq 0 \\ (-1)^{n-k} \binom{-k-1}{n-k} & \text{if } k \leq n \\ 0 & \text{otherwise} \end{cases} \quad (2.1)$$

Proof. When $k \geq 0$ the numerator and second denominator gamma functions in (1.1) have nonpositive integer argument. These two gamma functions are substituted by (1.3) with $s = 0$, $a = -n$ and $b = -n+k$, yielding the first case of (2.1). When $k \leq n$ the numerator and first denominator gamma functions in (1.1) have nonpositive integer argument. These two gamma functions are substituted by (1.3) with $s = 0$, $a = -n$ and $b = -k$, yielding the second case of (2.1). When $n < k < 0$ all three gamma functions in (1.1) have nonpositive integer argument. In this case any of the above substitutions leaves only one denominator gamma function with nonpositive integer argument, which is infinite, thus yielding the third case of (2.1). \square

Using a different method this was proved earlier in literature [5]. When taking s in (1.3) to be a small complex deviation δ from the integer arguments n and/or k , for each case the following identities result:

$$\binom{n+\delta}{k} = (-1)^k \binom{-(n+\delta)+k-1}{k} \quad (2.2)$$

$$\binom{n+\delta}{k+\delta} = (-1)^{n-k} \binom{-(k+\delta)-1}{n-k} \quad (2.3)$$

The right side binomial coefficients of these two identities have near nonnegative integer arguments and are therefore continuous. The left side binomial coefficients are therefore also continuous at these integer arguments in one direction in complex two-dimensional x, y argument space. Since it is also continuous in x, y argument space excluding these integer arguments, it is continuous at these integer arguments in all (except two) directions. The binomial coefficient defined by (1.1), (1.2) and (2.1) is therefore continuous at all complex (including all integer) arguments, except for negative integer x and non-integer y where it is infinite.

3 Agreement with some Other Identities

There are some identities which should remain valid under (2.1). The symmetry identity [3, 4] and the trinomial revision identity [3, 4] are valid for all complex x, y, z :

$$\binom{x}{y} = \binom{x}{x-y} \quad (3.1)$$

$$\binom{x}{y} \binom{y}{z} = \binom{x}{z} \binom{x-z}{y-z} \quad (3.2)$$

These two identities follow directly from definition (1.1), and remain valid under (2.1) [5]. The absorption identity [3, 4] is valid for all complex x, y (except $y = 0$, see below):

$$\binom{x}{y} = \frac{x}{y} \binom{x-1}{y-1} \quad (3.3)$$

The addition identity [3, 4] is valid for all complex x, y (except $x = y = 0$, see below and [5]):

$$\binom{x}{y} = \binom{x-1}{y} + \binom{x-1}{y-1} \quad (3.4)$$

These two identities follow from definition (1.1) and gamma function property (1.6), and remain valid under (2.1) with the exceptions mentioned. These exceptions follow from the fact that gamma function property (1.6) with $s = 0$ yields $1 = 0 \cdot \infty$, which when true is non-associative. This explains that in terms of gamma functions the absorption identity (3.3) is always true, but in terms of binomial coefficients it is always true except for $y = 0$. When the binomial coefficients in the addition identity (3.4) are substituted with gamma functions, and the common factor is eliminated using the gamma function property (1.6), then the following equation results:

$$\frac{x}{y(x-y)} \frac{\Gamma(x)}{\Gamma(y)\Gamma(x-y)} = \left[\frac{1}{y} + \frac{1}{x-y} \right] \frac{\Gamma(x)}{\Gamma(y)\Gamma(x-y)} \quad (3.5)$$

which is trivially true when $x \neq 0$ or $y \neq 0$. For the non-trivial case this expression should be evaluated in the limit to $x = y = 0$. From the gamma function property

(1.6) follows that for nonnegative integer n at $x = 0$:

$$\frac{1}{\Gamma(x-n)} = (-1)^n n! x + O(x^2) \quad (3.6)$$

which means that the following may be substituted at $x = y = 0$:

$$\frac{\Gamma(x)}{\Gamma(y)\Gamma(x-y)} = \frac{y(x-y)}{x} \quad (3.7)$$

The left side of (3.5) reduces to 1, and the right side becomes:

$$\left[\frac{1}{y} + \frac{1}{x-y} \right] \frac{y(x-y)}{x} = \frac{x-y}{x} + \frac{y}{x} \quad (3.8)$$

The right side of this equation evaluates to 1, except when the limit to $x = y = 0$ is taken for both terms separately, because then for both terms $0/0 = 1$ and the sum evaluates to 2. This is a consequence of the fact that addition and division are non-associative for infinitely small numbers. This explains that in terms of gamma functions the addition identity (3.4) is always true, but in terms of binomial coefficients it is always true except for $x = y = 0$.

4 Agreement with the Binomial Theorem

The binomial theorem for nonnegative integer power $[1, 2]$ defines the binomial coefficients of nonnegative integer arguments in terms of a finite series, which is the Taylor expansion of $x + y$ to the power n in terms of x at $x = 0$.

For nonnegative integer n and complex x, y :

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} y^{n-k} x^k \quad (4.1)$$

Changing all k into $n - k$ in the summation term reverses the order of summation and yields this identity with x and y interchanged, which is the Taylor expansion in terms of y at $y = 0$. Therefore for nonnegative integer n the Taylor expansions in terms of x or y are identical finite series. For negative integer power n there is the Taylor expansion in terms of x at $x = 0$.

For negative integer n and complex x, y :

$$(x + y)^n = \sum_{k=0}^{\infty} (-1)^k \binom{-n+k-1}{k} y^{n-k} x^k \quad (4.2)$$

In this identity appear the binomial coefficients of the first case of (2.1). This series converges only when $|x| < |y|$. The binomial coefficients of the second case of (2.1)

yield a different identity.

For negative integer n and complex x, y :

$$\begin{aligned}(x+y)^n &= \sum_{k=n}^{-\infty} (-1)^{n-k} \binom{-k-1}{n-k} y^{n-k} x^k \\ &= \sum_{k=0}^{\infty} (-1)^k \binom{-n+k-1}{k} x^{n-k} y^k\end{aligned}\tag{4.3}$$

This identity is (4.2) with x and y interchanged, and therefore the Taylor expansion in terms of y at $y = 0$. This series converges only when $|x| > |y|$. For negative integer n the Taylor expansions in terms of x or y are two different infinite series. When taking $y = 1$ in (4.1) to (4.3), it appears that for nonnegative integer n there is one finite series for any x , but for negative integer n there are two infinite series, one converging only when $|x| < 1$ and one only when $|x| > 1$. These two series are represented by the two cases in identity (2.1).

5 Conclusion

The binomial coefficient identities (1.1), (1.2) and (2.1) define the binomial coefficient as a continuous function for all complex (including all integer) arguments, except for negative integer x and non-integer y , in which case the binomial coefficient is infinite. This definition is in agreement with the binomial theorem. With this definition the identities (3.1) to (3.4) are always true in terms of gamma functions, albeit with the exceptions mentioned in (3.3) and (3.4) in terms of binomial coefficients. When using (3.3) or (3.4), for example in the derivation of combinatorial identities or other applications, it is important to be aware of these exceptions, although they may involve only binomial coefficients with one or two negative arguments and may therefore not be a problem in practical cases. Because continuity and symmetry may be more important than these exceptions, this definition may be preferred above other definitions.

References

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